Mathematical and Physical Basis of Medical Biophysics

Lecture 2

Mathematics Necessary for Understanding Physics. Physical Quantities and Units. Kinematics

10th September 2021

Gergely AGÓCS

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How to Get Prepared?

- university = **autonomous learning**

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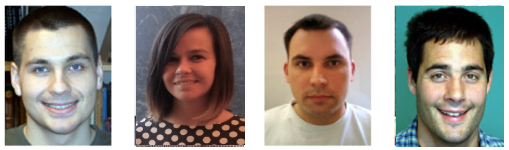
- sources:
  - **your** notes made in the lectures; ***only in the first four weeks***



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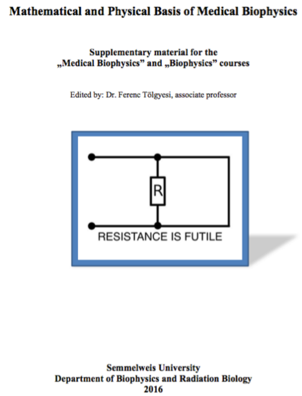
  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)



Mathematical and Physical Basis of Medical Biophysics

Supplementary material for the „Medical Biophysics" and „Biophysics" courses

Edited by: Dr. Ferenc Tölgyesi, associate professor

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Department of Biophysics and Radiation Biology
2016

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  - Tölgyesi: *Mathematical and Physical Basis of Medical Biophysics* (2016)
  - on-line material: https://itc.semmelweis.hu/moodle/course/view.php?id=313
    - subject requirements
    - lecture schedule and slides
    - textbook

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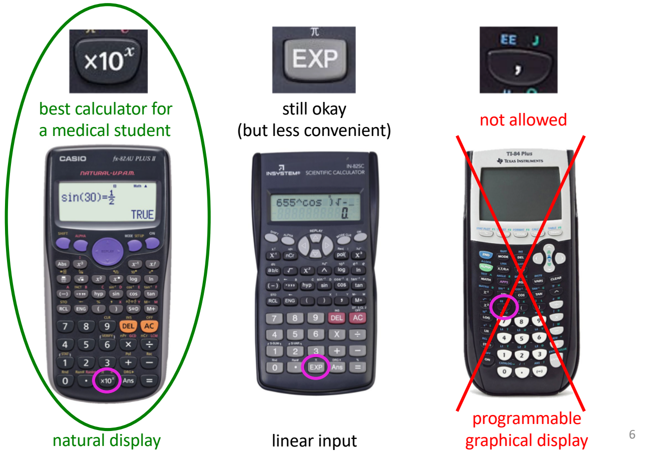
How to Use Scientific Notation?



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Use of Symbols in Science

In science we use a large number of **Latin** and **Greek** letters (and their combinations) as symbols, so it is inevitable to learn the Greek alphabet.

However, the number of quantities and units is much greater than the number of available letters, and this can lead to confusion. Your help: CONTEXT

ambiguity! ← **BEWARE!** → **inconsistency!**

ambiguity! for example: **c and C**
- a general constant
- Celsius
- carbon
- capacitance and capacitor
- coulomb
- centi-
- concentration (many different kinds)
- speed of sound
- speed of light
- specific heat capacity

inconsistency! for example:
- density ← ***ρ*** [rho], ***d***
- speed ← ***v***, ***c***
- multiplication ← **×**, **·**, *****
- frequency ← ***f***, ***ν*** [nu]
- proportionality ← **∝**, **~**

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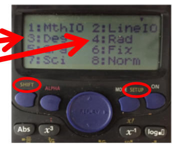
Angles


D: degrees mode


R: radians mode

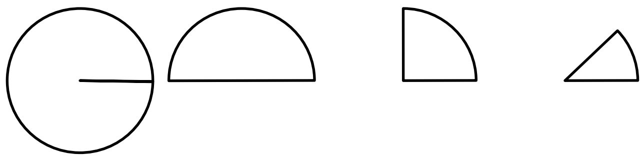
– shift
– setup
– 3 (for degrees)
– 4 (for radians)



revolution: one turn
degree: practical, traditional unit
radian: scientific unit, arc/radius

1 revolution = 360° = 2π rad

1° = 60′ = 3600″



one **revolution**	half **revolution**	quarter **revolution**	1/8 **revolution**
360° **degrees**	180° **degrees**	90° **degrees**	45° **degrees**
2π **radians**	π **radian**	π/2 **radian**	π/4 **radian**

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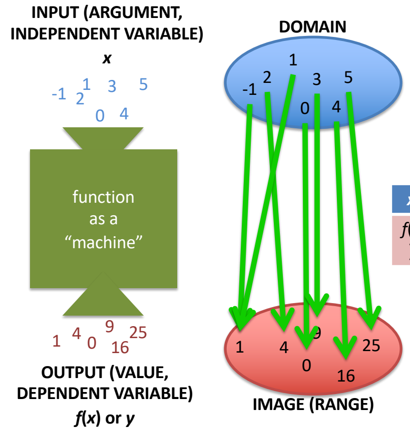
What is a Function?

Unambiguous assignment of one set of values to an other set of values



$x \mapsto f(x)$ or $y = f(x)$

x	−1	0	1	2	3	4	5
$f(x)$	1	0	1	4	9	16	25

$x \mapsto f(x)$ or $y = f(x)$

f is the function defining the relationship between x and $f(x)$

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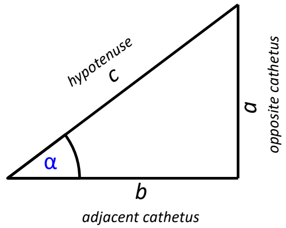
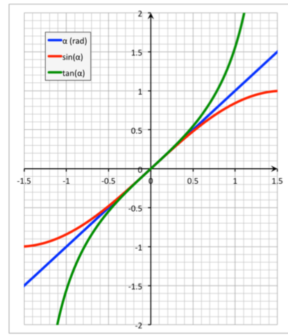
Trigonometric Functions

degree: practical, traditional unit
radian: scientific unit, arc/radius
1 revolution = 360° = 2π rad



sine: $\sin(\alpha) = a/c$
cosine: $\cos(\alpha) = b/c$
tangent: $\tan(\alpha) = \mathrm{tg}(\alpha) = a/b$

for small angles (<10° ≈ 0.2 rad):
$\sin(\alpha) \approx \alpha\ [\mathrm{rad}] \approx \tan(\alpha)$



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Linear Function

INTEGRAL FORM

VARIABLES: dependent variable; independent variable

$y = a \cdot x + b$

PARAMETERS: slope (gradient, increment); y-axis intercept

"DIFFERENTIAL" FORM

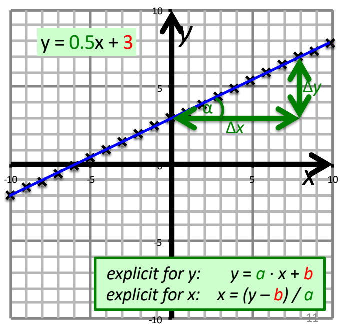
$\Delta y \propto \Delta x$

The **change** of the dependent variable is proportional to the **change** of the independent variable

if $x = 0$ then $y = b$

if $\Delta x = 1$ then $\Delta y = a$

$a = \Delta y/\Delta x = \tan\alpha$



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Linear Function: Some Examples
from the Biophysics Formula Collection

#1: The ideal gas law (I.35)
$pV = nRT$ (if n & V are constant)
$p = nR/V \cdot T + 0$
$y = a \cdot x + b$

#2: Photoelectric effect (II.37)
$E_{kin} = hf - W_{em}$
$E_{kin} = h \cdot f + (-W_{em})$
$y = a \cdot x + b$

#3: Attenuation coefficient (II.85)
$\mu = \mu_m \cdot \rho$
$\mu = \mu_m \cdot \rho + 0$
$y = a \cdot x + b$

#4: Ohm's law
$R = U/I$
$I = 1/R \cdot U + 0$
$y = a \cdot x + b$

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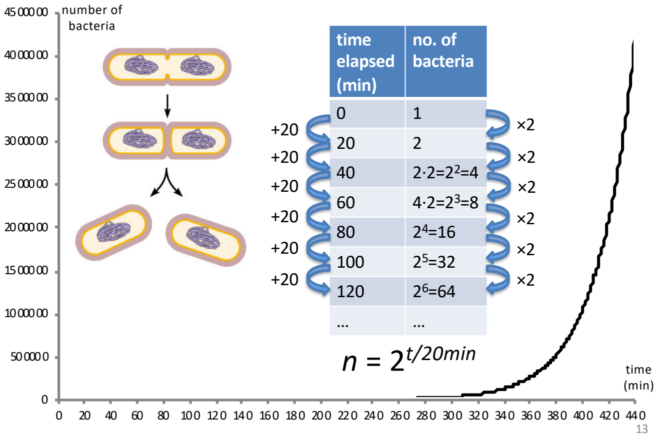
Exponential Function: Example #1



	time elapsed (min)	no. of bacteria	
	0	1	
+20	20	2	×2
+20	40	$2 \cdot 2 = 2^2 = 4$	×2
+20	60	$4 \cdot 2 = 2^3 = 8$	×2
+20	80	$2^4 = 16$	×2
+20	100	$2^5 = 32$	×2
+20	120	$2^6 = 64$	×2
	…	…	

$$n = 2^{t/20min}$$

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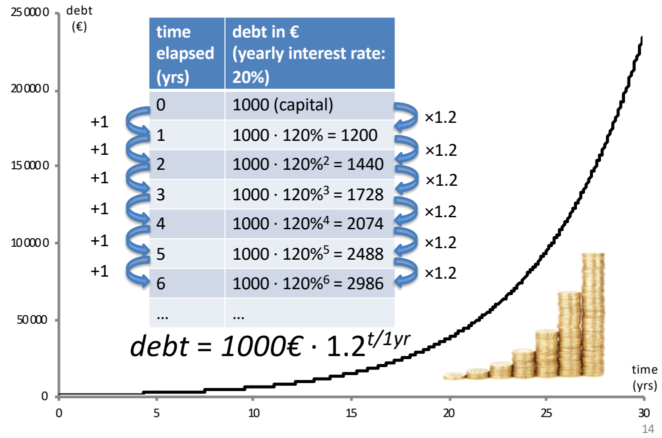
Exponential Function: Example #2



	time elapsed (yrs)	debt in € (yearly interest rate: 20%)	
	0	1000 (capital)	
+1	1	1000 · 120% = 1200	×1.2
+1	2	1000 · 120%^{2} = 1440	×1.2
+1	3	1000 · 120%^{3} = 1728	×1.2
+1	4	1000 · 120%^{4} = 2074	×1.2
+1	5	1000 · 120%^{5} = 2488	×1.2
+1	6	1000 · 120%^{6} = 2986	×1.2
	…	…	

$$debt = 1000€ \cdot 1.2^{t/1yr}$$

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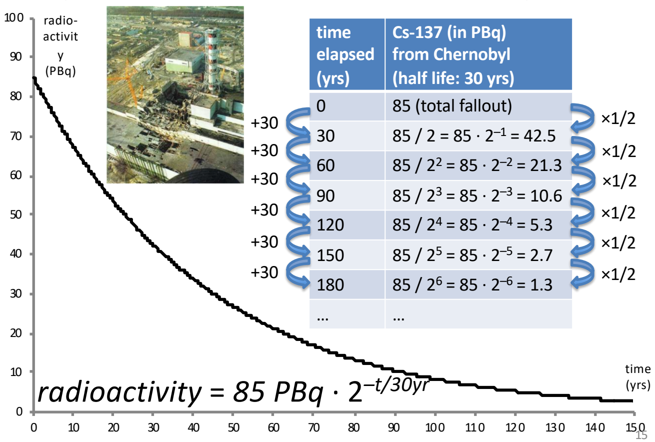
Exponential Function: Example #3



	time elapsed (yrs)	Cs-137 (in PBq) from Chernobyl (half life: 30 yrs)	
	0	85 (total fallout)	
+30	30	$85 / 2 = 85 \cdot 2^{-1} = 42.5$	×1/2
+30	60	$85 / 2^2 = 85 \cdot 2^{-2} = 21.3$	×1/2
+30	90	$85 / 2^3 = 85 \cdot 2^{-3} = 10.6$	×1/2
+30	120	$85 / 2^4 = 85 \cdot 2^{-4} = 5.3$	×1/2
+30	150	$85 / 2^5 = 85 \cdot 2^{-5} = 2.7$	×1/2
+30	180	$85 / 2^6 = 85 \cdot 2^{-6} = 1.3$	×1/2
	…	…	

$$radioactivity = 85\ PBq \cdot 2^{-t/30yr}$$

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Exponential Function

INTEGRAL FORM

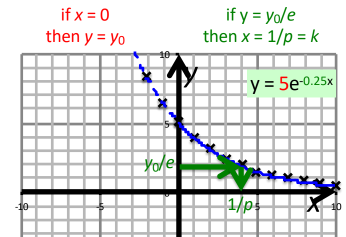
$$y = b \cdot a^x$$

PRACTICAL MODIFICATIONS:
- the base number is preferred to be *e*
- a new factor parameter *p* (or 1/*k*) is necessary in the exponent
- use a negative sign in the exponent
- *b* is rather denoted by y_0

VARIABLES: dependent variable, independent variable

$$y = y_0 \cdot e^{-px} = y_0 \cdot e^{-x/k}$$

PARAMETERS: pre-exponential coefficient, exponential coefficient



explicit for y: $y = y_0 \cdot e^{-px}$
explicit for x: $x = \ln(y / y_0) / (-p)$

"DIFFERENTIAL" FORM

$$\Delta y/y \propto \Delta x$$

The **relative change** of the dependent variable is proportional to the **change** of the independent variable

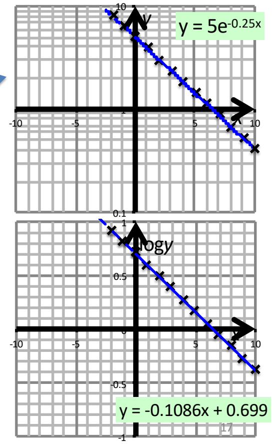
Exponential Function: Linearization

graphical linearization
plot y on a log scale as a function of x:
*the relationship **looks** linear but it **is** still exponential*

INTEGRAL FORM

$y = y_0 \cdot e^{-px}$

$$\log y = \log\left(y_0 \cdot e^{-p\cdot x}\right)$$
$$\log y = \log y_0 + \log\left(e^{-p\cdot x}\right)$$
$$\log y = \log y_0 - p\cdot x\cdot \log e$$
$$\underbrace{\log y}_{y} = \underbrace{-p\cdot \log e}_{a}\cdot x + \underbrace{\log y_0}_{b}$$



intercept = log(y_0)
log(5) = 0.699
slope = – p·log(e)
–0.25·log(e) = –0.1086

arithmetical linearization
plot log(y) as a function of x:
*the relationship **is** linear*

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Exponential Function: Some Examples

from the Biophysics Formula Collection

#1: Law of radiation attenuation (II.11)

$J = J_0 \cdot e^{-\mu x}$

#2: Boltzmann's distribution (I.25)

$n_i = n_0 \bullet e^{-\Delta\varepsilon/(kT)}$

#3: Decay law (II.96)

$N = N_0 \bullet e^{-\lambda t}$

#4: Discharging an RC circuit (VII.2)

$U = U_0 \bullet e^{-t/(RC)}$

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$y = y_0 \cdot e^{-px}$

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$n_i = n_0 \bullet e^{-\Delta\varepsilon/(kT)}$

$y = y_0 \cdot e^{-x/k}$

#3: Decay law (II.96)

$N = N_0 \bullet e^{-\lambda t}$

$y = y_0 \cdot e^{-px}$

#4: Discharging an RC circuit (VII.2)

$U = U_0 \bullet e^{-t/(RC)}$

$y = y_0 \cdot e^{-x/k}$

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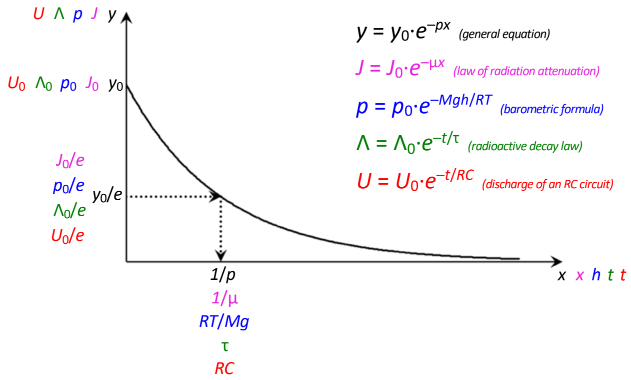
Graph of Exponential Functions

from the Biophysics Formula Collection



$y = y_0 \cdot e^{-px}$ *(general equation)*

$J = J_0 \cdot e^{-\mu x}$ *(law of radiation attenuation)*

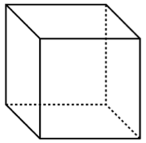
$p = p_0 \cdot e^{-Mgh/RT}$ *(barometric formula)*

$\Lambda = \Lambda_0 \cdot e^{-t/\tau}$ *(radioactive decay law)*

$U = U_0 \cdot e^{-t/RC}$ *(discharge of an RC circuit)*

Power Function: Example

mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2



Power Function

INTEGRAL FORM

VARIABLES: dependent variable — independent variable

$y = b \cdot x^a$

PARAMETERS: pre-exponential coefficient — exponent

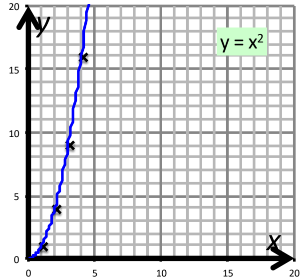
explicit for y: $y = b \cdot x^a$
explicit for x: $x = (y / b)^{1/a}$

"DIFFERENTIAL" FORM

$\Delta y/y \propto \Delta x/x$

The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

if $x = 1$
then $y = b$



inverse proportionality and **square root** functions are also power functions

$y = \frac{b}{x} = b \cdot x^{-1}$

$y = \sqrt{x} = x^{\frac{1}{2}}$

Power Function: Linearization

graphical linearization
plot both *y* and x on log scales:
*the relationship **looks** linear but it **is** still power function*

INTEGRAL FORM

$y = b \cdot x^a$

$\log y = \log(b \cdot x^a)$

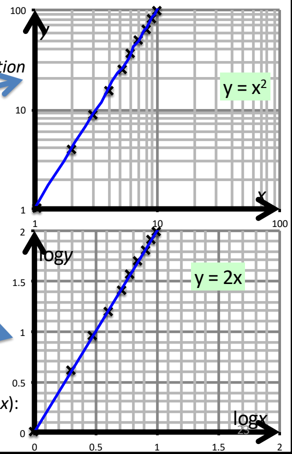
$\log y = \log b + \log(x^a)$

$\log y = \log b + a \cdot \log x$

$\underbrace{\log y}_{y} = a \cdot \underbrace{\log x}_{x} + \underbrace{\log b}_{b}$

intercept = log*b*
log1 = 0
slope = *a*
a = 2

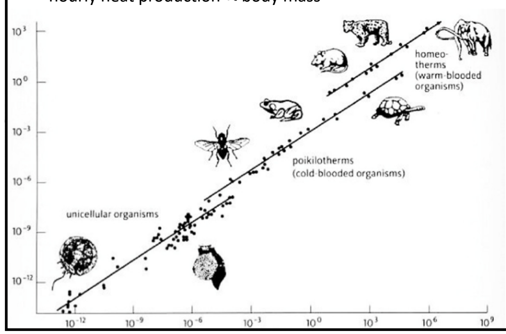
arithmetical linearization
plot log(*y*) as a function of log(*x*):
*the relationship **is** linear*



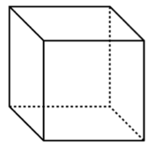
Power Function: Example

Allometric scaling
(E.g. Kleiber's law)

hourly heat production ∝ body mass$^{3/4}$



mass ∝ volume ∝ [body]length^3
surface area ∝ [body]length^2



Power Function: Some Examples

from the Biophysics Formula Collection

#1: The de Broglie wavelength (I.3)

$\lambda = h/p$

$\lambda = h \cdot p^{-1}$

#2: Stefan–Boltzmann law (II.41)

$M_{\text{black}} = \sigma \cdot T^4$

#3: Duane–Hunt law (II.80)

$$\lambda_{\min} = \frac{hc}{eU_{\text{anode}}}$$

$\lambda_{\min} = hc/e \cdot U^{-1}$

#4: Mass dependence of eigenfrequency (Resonance 6)

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$f_0 = k^{1/2}/(2\pi) \cdot m^{-1/2}$

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Power Function: Some Examples

from the Biophysics Formula Collection

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$y = b \cdot x^a$

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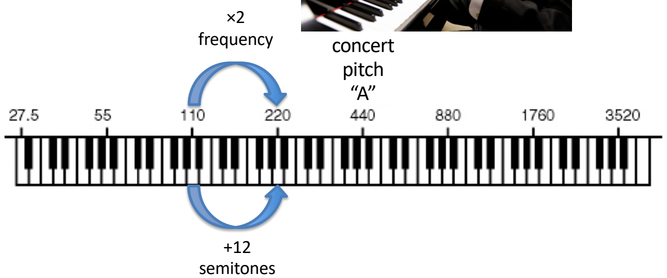
Logarithmic Function: Example



Logarithmic Function

INTEGRAL FORM

$y = b \cdot \log_a(x)$

PRACTICAL CONSIDERATIONS:

- base is 10 (sometimes *e* or 2)
- if the base is fixed this will modify the factor paramater according to the following identity:

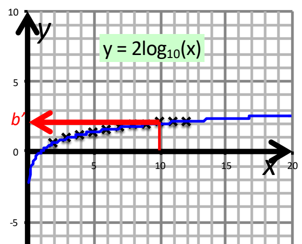
$b \cdot \log_a(x) = b/\log_{10}(a) \cdot \log_{10}(x) = b' \cdot \log_{10}(x)$

VARIABLES: dependent variable; independent variable

$y = b' \cdot \log_{10}(x)$

PARAMETERS: factor parameter

if $x = 10$
then $y = b'$



„DIFFERENTIAL" FORM

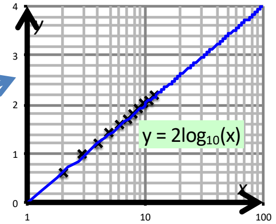
$\Delta y \sim \Delta x/x$

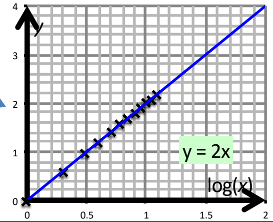
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable

Logarithmic Function: Linearization

graphical linearization
plot y on lin and x on log scales:
*the relationship **looks** linear but it **is** still a log function*

INTEGRAL FORM
$y = b' \cdot \log_{10}(x)$





arithmetical linearization
plot y as a function of log(x):
*the relationship **is** linear*

Logarithmic Function: Some Examples

from the Biophysics Formula Collection ...and elsewhere

#1: The statistical definition of entropy
(III.72)
$S = k \ln \Omega$
$S = k \cdot \log_e(\Omega)$

#2: The decibel (dB) scale
(VII.10)
$n = 10 \log A_p$
$n = 10 \cdot \log_{10}(A_p)$

#3: The definition of absorbance
(VI.34)
$A = \lg(J_0/J)$
$A = 1 \cdot \log_{10}(J_0/J)$

#4: The pH scale
$\mathrm{pH} = -\log[\mathrm{H}^+]$
$\mathrm{pH} = -1 \cdot \log_{10}([\mathrm{H}^+]/(1\ \mathrm{M}))$

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Logarithmic Function: Some Examples

from the Biophysics Formula Collection ...and elsewhere

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$S = k \ln \Omega$
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$\mathrm{pH} = -\log[\mathrm{H}^+]$
$\mathrm{pH} = -1 \cdot \log_{10}([\mathrm{H}^+]/(1\ \mathrm{M}))$
$y = b \cdot \log_a(x)$

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Functions

Summary

LINEAR FUNCTION	EXPONENTIAL FUNCTION
$\Delta y \sim \Delta x$	$\Delta y/y \sim \Delta x$
The **change** of the dependent variable is proportional to the **change** of the independent variable	The **relative change** of the dependent variable is proportional to the **change** of the independent variable
y* vs. *x	**log*y* vs. *x***

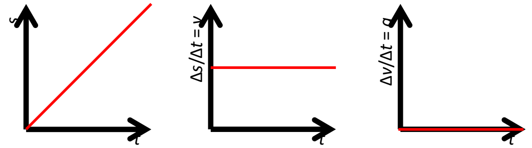
Linearization

y* vs. log*x	**log*y* vs. log*x***
LOGARITHMIC FUNCTION	POWER FUNCTION
$\Delta y \sim \Delta x/x$	$\Delta y/y \sim \Delta x/x$
The **absolute change** of the dependent variable is proportional to the **relative change** of the independent variable	The **relative change** of the dependent variable is proportional to the **relative change** of the independent variable

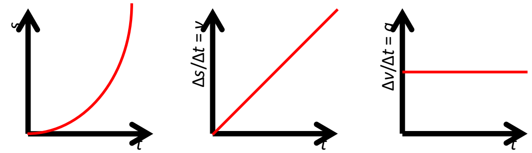
Derivative and Integral: Application

Rectilinear Motion

uniform rectilinear motion:



uniform rectilinear acceleration:



Circular Motion

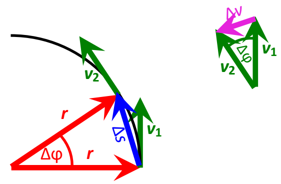
Quantities, Units, and Equation

angular displacement: $\Delta\varphi = \varphi_2 - \varphi_1$ — $[\Delta\varphi]$ = rad

angular velocity, angular frequency: $\omega = \Delta\varphi/\Delta t$ — $[\omega]$ = rad/s

tangential velocity: $v = r\cdot\Delta\varphi/\Delta t = r\cdot\omega$ — $[v]$ = m/s

centripetal acceleration: $a_{cp} = v^2/r = r\cdot\omega^2$ — $[a]$ = m/s^2



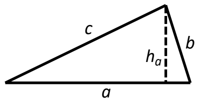
(1) approximation in case of small angles:
displacement = arc length = $v\cdot\Delta t \approx \Delta s$

(2) due to similarity:
$\Delta v/v = \Delta s/r$

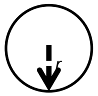
(1) + (2):
$\Delta v/v = v\cdot\Delta t/r$

$a_{cp} = v^2/r$

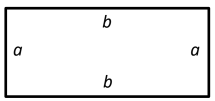
Perimeter & Area



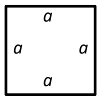
TRIANGLE
perimeter: $a+b+c$
area: $a*h_a/2$



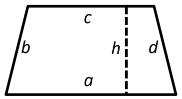
CIRCLE
perimeter: $2r\pi$
area: $r^2\pi$



RECTANGLE
perimeter: $2*(a+b)$
area: $a*b$

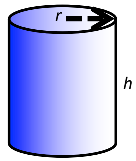


SQUARE
perimeter: $4a$
area: $a*a=a^2$



TRAPEZOID
perimeter: $a+b+c+d$
area: $(a+c)/2*h$

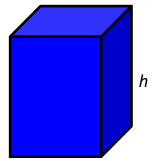
Surface & Volume



CYLINDER (open)

surface (wall only):
$2r\pi*h$

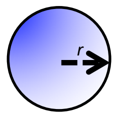
volume: $r^2\pi*h$



PRISM (open)

surface (wall only):
(perimeter of base)*h

volume: (area of base)*h



SPHERE

surface:
$4r^2\pi$

volume: $4r^3\pi/3$

Units – SI Base & Derived Units

physical quantity	symbol	unit	symbol
length	*l, x, s, d*	meter	m
mass	*m*	kilogram	kg
time	*t*	second	s
temperature	*T*	kelvin	K
electric current	*I*	ampere	A
amount of substance	*n, N,* ν [nu]	mole	mol
luminous intensity	I_v	candela	cd

The SI base units

physical quantity	symbol	unit	symbol	derivation
speed	*v, c*	–	–	$m \cdot s^{-1}$
acceleration	*a*	–	–	$m \cdot s^{-2}$
force	*F*	newton	N	$kg \cdot m \cdot s^{-2}$
energy	*E*	joule	J	$kg \cdot m^2 \cdot s^{-2}$
power	*P*	watt	W	$kg \cdot m^2 \cdot s^{-3}$
intensity	*I*	–	–	$kg \cdot s^{-3}$
pressure	*p*	pascal	Pa	$kg \cdot m^{-1} \cdot s^{-2}$

Some SI derived units

Units – SI Prefixes

prefix	symbol	meaning	etymology
exa	E	$\times 10^{18} = \times 1000^6$	Greek 6 (ἕξ = hex)
peta	P	$\times 10^{15} = \times 1000^5$	Greek 5 (πέντε = pente)
tera	T	$\times 10^{12} = \times 1000^4$	Greek 4 (τέτταρες = tettares), originally: monster (τέρας = teras)
giga	G	$\times 10^{9} = \times 1000^3$	Greek giant (γίγας = gigas)
mega	M	$\times 10^{6} = \times 1000^2$	Greek great (μέγας = megas)
kilo	k	$\times 10^{3} = \times 1000^1$	Greek 1000 (χίλιοι = khilioi)
hekto	h	$\times 10^{2}$	Greek 100 (ἑκατόν = hekaton)
deca	da (dk)	$\times 10^{1}$	Greek 10 (δέκα = deka)
deci	d	$\times 10^{-1}$	Latin 10 (decem)
centi	c	$\times 10^{-2}$	Latin 100 (centum)
milli	m	$\times 10^{-3} = \times 1000^{-1}$	Latin 1000 (mille, *pl.* milia)
micro	μ	$\times 10^{-6} = \times 1000^{-2}$	Greek small (μικρός = mikros)
nano	n	$\times 10^{-9} = \times 1000^{-3}$	Greek dwarf (νᾶνος = nanos)
pico	p	$\times 10^{-12} = \times 1000^{-4}$	Spanish small, bit (pico)
femto	f	$\times 10^{-15} = \times 1000^{-5}$	Danish 15 (femten)
atto	a	$\times 10^{-18} = \times 1000^{-6}$	Danish 18 (atten)

Units – Conversion

from "with prefix" to "no prefix":
15 km = 15 · 10^3 m
15 cg = 15 · 10^{-2} g

from "no prefix" to "with prefix":
15 m = 15 / 10^3 km
15 g = 15 / 10^{-2} cg

from "with prefix" to "with prefix":
15 km = 15 · 10^3 m = 15 · 10^3 / 10^{-2} cm

when the unit has an exponent:
15 km^3 = 15 · $(10^3\ m)^3$ = 15 · $(10^3)^3$ m^3
15 m^3 = 15 / $(10^3)^3$ km^3

liters to and from cubic meters:
1 m^3 = 10 hL = 1000 L
1 dm^3 = 1 L
1 cm^3 = 1 mL
1 mm^3 = 1 μL

time to seconds:
2 days 3 h 12 min 30 s = ((2·24+3)·60+12)·60+30 s

degrees, minutes of arc, seconds of arc:
45° 40′ 30″ = $(45+40/60+30/60^2)$°

degrees to and from radians:
1 rad = (360/2π)°
1° = (2π/360) rad

compound units:
15 kg/m^3 = 15 · 10^3 / (1/$(10^{-2})^3$) g/cm^3
45 km/h = 45 · 10^3 / 3600 m/s

degrees Celsius to and from kelvins:
T = 15 °C = (15+273) K
T = 15 K = (15–273) °C
ΔT = 15 °C = 15 K
ΔT = 15 K = 15 °C